Math Challenge

2017

Team Theme

This is a TEAM event. You have 45 minutes for this event. You are to work on the Team Theme as a group. You may split up the team theme problems in any way you choose; however, use one copy to write your final solutions (each team has been given 3 copies), and be sure to put your TEAM NUMBER on each page (your team number is listed on the back of your schedule sheet). Do not use your school name.

If you need extra space to answer a particular problem, you may use the backside of the problems or attach more than one copy of the solution sheets. If you do use a backside or attach another copy of the solution sheets, please make a note so the grader knows where to look.

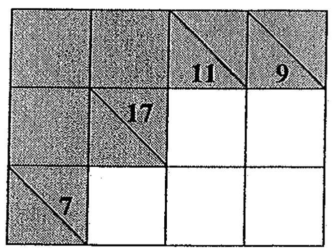
Write all answers in mathematically and grammatically complete sentences. You will be graded not only on the accuracy of your answers, but also on how well your team explains itself mathematically, and on the creativity in your narrative. It is your job to convince the graders that you understand the problem completely and to make your presentation interesting.

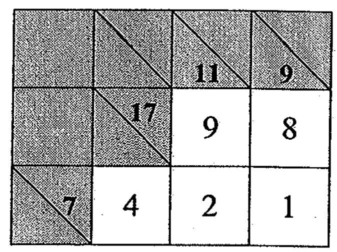
I. Kakuro Puzzles

Kakuro puzzles are similar with crosswords, but instead of letters the board is filled with digits from 1 to 9. Taken together, all of the blank squares that are adjacent without interruption – in a given row or column are called a **word** (or a **run**).

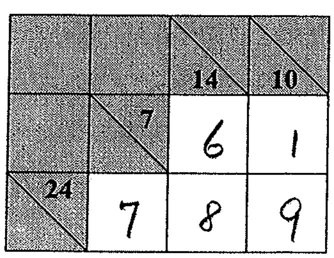
- There is a **clue** for every word – the clue tells you what the ***sum of the terms in the word*** must be.
- Words can only use the digits 1 through 9, and a digit can be used only once in a word.

Consider the Kakuro puzzle below for example. The clue for the first row is 17 and there is only one way to write 17 using *two* distinct digits, $17 = 9 + 8$. The clue for the second row is 7 and there is only one way to write 7 using *three* distinct digits, $7 = 4 + 2 + 1$. The solution to the puzzle is shown on the right. You can try to rearrange these digits but will soon find that this is the only possibility.





1. Solve the Kakuro puzzle below.



2. Solve the Kakuro puzzle below.

			10\	17\
	7\	11\12	3	9
\30	6	9	7	8
\3	1	2		

3. Solve the Kakuro puzzle below.

			23\	11\
	6\	12\8	6	2
\28	4	7	8	9
\16	2	5	9	

II. Partitions of Numbers

You probably have noticed in Kakuro puzzles that a given number (a clue) can be broken up in a few ways. But if a given number 8 must be broken up for a word with three digits, then there are two possibilities (using different addends); $8 = 5 + 2 + 1 = 4 + 3 + 1$. How many different possible combinations a given number can take is an important mathematical topic. We call $8 = 5 + 2 + 1$ a ***partition*** of the number 8. Note that $8 = 8$ is also considered a partition (using 1 addend).

Partitions were first studied systematically by the great Swiss mathematician **Leonhard Euler**.

1. How many ways can the number 7 be written as a sum of *different* positive integers? List and write them in each column. Write "***none***" if it is not possible.

1 addend	2 addends	3 addends	4 addends
7	6 + 1 5 + 2 4 + 3	4 + 2 + 1	none

Total number of ways = 5

2. If you wrote ***none*** in any column, explain clearly why it is not possible to have a partition with that many integers.

The smallest possible sum of 4 distinct positive integers is

$1+2+3+4 = 10$. But $10 > 7$.

3. Multiply the following binomials and simplify; $(1+x)\cdot(1+x^2)\cdot(1+x^3)\cdot(1+x^4)\cdot(1+x^5)\cdot(1+x^6)\cdot(1+x^7)$. Write only the final simplified answer with the exponents in descending order.

$$x^{28} + x^{27} + x^{26} + 2x^{25} + 2x^{24} + 3x^{23} + 4x^{22} + 5x^{21} + 5x^{20}$$

$$+ 6x^{19} + 7x^{18} + 7x^{17} + 8x^{16} + 8x^{15} + 8x^{14} + 8x^{13} + 8x^{12} + 7x^{11} + 7x^{10}$$

$$+ 6x^9 + 5x^8 + \boxed{5x^7} + 4x^6 + 3x^5 + 2x^4 + 2x^3 + x^2 + x + 1$$

4. In the result of #3 above, what should be the largest exponent? Explain your answer.

$\boxed{28}$

$$x^1 \cdot x^2 \cdot x^3 \cdot x^4 \cdot x^5 \cdot x^6 \cdot x^7 = x^{1+2+3+4+5+6+7} = x^{28}$$

5. In the result of #3 above, what is the coefficient of the x^7-term?

$\boxed{5}$

6. In an algebra class you have studied the properties of exponents; $x^{m+n} = x^m \cdot x^n$. Thus, we can write x^9 as $x^9 = x^{5+4} = x^5 \cdot x^4$ or $x^9 = x^{6+2+1} = x^6 \cdot x^2 \cdot x$.
In how many different ways can x^7 be written as the product of distinct positive powers of x? Write and list them.

$x^7 = x^7$; $x^7 = x^6 \cdot x$; $x^7 = x^4 \cdot x^2 \cdot x$

$x^7 = x^5 \cdot x^2$

$x^7 = x^4 \cdot x^3$

7. How many ways can the number **10** be written as a sum of *different* positive integers? List and write them in each column. Write "***none***" if it is not possible.

1 addend	2 addends	3 addends	4 addends	5 addends
10	9+1 8+2 7+3 6+4	7+2+1 6+3+1 5+4+1 5+3+2	4+3+2+1	None

Total number of ways = 10

8. If you wrote *none* in any column, explain clearly why it is not possible to have a partition with that many integers.

The smallest possible sum of 5 distinct positive integers is $1+2+3+4+5=15$. But $15>10$.

9. Multiply the following binomials and simplify. You may use the result from #3.
$(1+x)\cdot(1+x^2)\cdot(1+x^3)\cdot(1+x^4)\cdot(1+x^5)\cdot(1+x^6)\cdot(1+x^7)\cdot(1+x^8)\cdot(1+x^9)\cdot(1+x^{10})$
Write only the final simplified answer with the exponents in decreasing order.

$x^{55}+x^{54}+x^{53}+2x^{52}+2x^{51}+3x^{50}+4x^{49}+5x^{48}+6x^{47}+8x^{46}$
$+10x^{45}+11x^{44}+13x^{43}+15x^{42}+17x^{41}+20x^{40}+22x^{39}+24x^{38}+27x^{37}$
$+29x^{36}+31x^{35}+33x^{34}+35x^{33}+36x^{32}+38x^{31}+39x^{30}+39x^{29}+40x^{28}$
$+40x^{27}+39x^{26}+39x^{25}+38x^{24}+36x^{23}+35x^{22}+33x^{21}+31x^{20}+29x^{19}$
$+27x^{18}+24x^{17}+22x^{16}+20x^{15}+17x^{14}+15x^{13}+13x^{12}+11x^{11}+\boxed{10x^{10}}$
$+8x^{9}+6x^{8}+5x^{7}+4x^{6}+3x^{5}+2x^{4}+2x^{3}+x^{2}+x+1$

10. In the result of #9 above, what should be the largest exponent? Explain your answer.

$\boxed{55}$

$$x \cdot x^2 \cdot x^3 \cdot x^4 \cdot x^5 \cdot x^6 \cdot x^7 \cdot x^8 \cdot x^9 \cdot x^{10} = x^{1+2+3+4+5+6+7+8+9+10} = x^{55}$$

11. In the result of #9 above, what is the coefficient of the x^{10}-term?

$\boxed{10}$

12. In how many different ways can x^{10} be written as the product of distinct positive powers of x? Write and list them.

$x^{10} = x^{10}$; $x^{10} = x^9 \cdot x$; $x^{10} = x^7 \cdot x^2 \cdot x$; $x^{10} = x^4 \cdot x^3 \cdot x^2 \cdot x$

$x^{10} = x^8 \cdot x^2$ $\quad$ $x^{10} = x^6 \cdot x^3 \cdot x$

$x^{10} = x^7 \cdot x^3$ $\quad$ $x^{10} = x^5 \cdot x^4 \cdot x$

$x^{10} = x^6 \cdot x^4$ $\quad$ $x^{10} = x^5 \cdot x^3 \cdot x^2$

III. Count the Number of Ways

1. Determine the number of solutions to the equation $a + b + c = 7$ where a, b, c are integers such that $0 \le a \le 2$, $2 \le b \le 5$, and $1 \le c \le 4$. List the values of variables whose sum is 7. (A number may be used by more than one variables.)

a	0	0	0	1	1	1	1	2	2	2
b	3	4	5	2	3	4	5	2	3	4
c	4	3	2	4	3	2	1	3	2	1
Sum 7	7	7	7	7	7	7	7	7	7	7

2. If we multiply and simplify the following polynomials, what would be the coefficient of the x^7-term?
$(1+x+x^2)\cdot(x^2+x^3+x^4+x^5)\cdot(x+x^2+x^3+x^4)$

$\boxed{10}$

3. Justify your answer to #2 by writing x^7 in the form $x^7 = x^{a+b+c} = x^a \cdot x^b \cdot x^c$ and satisfies the equation and the conditions of #1.

$x^7 = x^0 \cdot x^3 \cdot x^4$; $x^7 = x \cdot x^2 \cdot x^4$; $x^7 = x^2 \cdot x^2 \cdot x^3$

$x^7 = x^0 \cdot x^4 \cdot x^3$; $x^7 = x \cdot x^3 \cdot x^3$; $x^7 = x^2 \cdot x^3 \cdot x^2$

$x^7 = x^0 \cdot x^5 \cdot x^2$; $x^7 = x \cdot x^4 \cdot x^2$; $x^7 = x^2 \cdot x^4 \cdot x$

$x^7 = x \cdot x^5 \cdot x$;

4. Determine the number of ways to obtain *65 ¢* if we have 3 nickels, 5 dimes, and 2 quarters. List the value of types of coins in the table below.

nickels	5	15	0	10	15
dimes	10	0	40	30	50
quarters	50	50	25	25	0
65 ¢	65	65	65	65	65

5. If we multiply and simplify the following polynomials, what would be the coefficient of the x^{65}-term?
$(1+x^5+x^{10}+x^{15})\cdot(1+x^{10}+x^{20}+x^{30}+x^{40}+x^{50})\cdot(1+x^{25}+x^{50})$

$\boxed{5}$

6. Justify your answer to #5 by writing x^{65} in the form $x^{65}=x^{n+d+q}=x^n\cdot x^d\cdot x^q$ and satisfies the conditions of #4.

$x^{65}=x^{0}\cdot x^{40}\cdot x^{25}$; $x^{65}=x^{5}\cdot x^{10}\cdot x^{50}$; $x^{65}=x^{10}\cdot x^{30}\cdot x^{25}$

$x^{65}=x^{15}\cdot x^{0}\cdot x^{50}$; $x^{65}=x^{15}\cdot x^{50}\cdot x^{0}$